

A NOTE ON THE CONTINUOUS SELF-MAPS OF THE LADDER SYSTEM SPACE

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ABSTRACT. We give a partial characterization of the continuous self-maps of the ladder system space K_S . Our results show that K_S is highly nonrigid. We also discuss reasonable notions of “few operators” for spaces $C(K)$ with scattered K and we show that $C(K_S)$ does not have few operators for such notions.

1. INTRODUCTION

The ladder system space K_S is a standard example of a compact scattered space of finite (Cantor–Bendixson) height ([1, pg. 164]). It was used by R. Pol [4] to obtain the first example of a weakly Lindelöf Banach space $C(K)$ that is not WCG. The space K_S is the one-point compactification $\omega_1 \cup \{\infty\}$ of the first uncountable ordinal ω_1 endowed with the ladder system topology τ_S (see Section 2). A fairly uninteresting type of continuous self-map ϕ of K_S consists of those that are “almost constant” in the sense that they have countable range (and thus yield “small” composition operators $\mathcal{C}_\phi : f \mapsto f \circ \phi$ on $C(K_S)$ of separable range). Those include the continuous maps that do not fix the point at infinity ∞ . One might wonder whether K_S is “almost rigid” in the sense that the continuous self-maps of K_S are either “almost constant” or are close to the identity, i.e., have lots of fixed points. We answer this question in the negative, providing a simple example of a continuous self-map of K_S having uncountable range and whose only fixed point is ∞ (Example 2.1). We will show that the existence of such a map already implies that $C(K_S)$ does not have few operators for some natural notions of “few operators” (Subsection 2.1). Even though maps of uncountable range may fix only the point at infinity, we prove that if $\phi : K_S \rightarrow K_S$ is a continuous map fixing ∞ then there must exist a club $F \subset \omega_1$ such that $\phi(\alpha) = \alpha$ or $\phi(\alpha) = \infty$ for all $\alpha \in F$ (Proposition 1).

Another natural question is whether a continuous self-map of K_S can move uncountably many limit ordinals while not mapping them to the infinity point. We answer the latter affirmatively, showing that given a club subset F of the limit ordinals $L(\omega_1)$ then on the nonstationary set $L(\omega_1) \setminus F$ one is reasonably free to choose the value of a continuous map (Theorem 2).

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2. MAIN RESULTS

We denote by $L(\omega_1)$ the subset of ω_1 consisting of limit ordinals and we set $S(\omega_1) = \omega_1 \setminus L(\omega_1)$. By a *ladder system* \mathcal{S} on ω_1 we mean a family $\mathcal{S} = (s_\alpha)_{\alpha \in L(\omega_1)}$ where, for each limit ordinal α , s_α is the image of a strictly increasing sequence $(s_\alpha^n)_{n \in \omega}$ in $S(\omega_1)$ order-converging to α . The *ladder system topology* $\tau_{\mathcal{S}}$ on ω_1 is the one for which the elements of $S(\omega_1)$ are isolated and the fundamental neighborhoods of a limit ordinal α are unions of $\{\alpha\}$ with sets cofinite in s_α . Then $S(\omega_1)$ is a discrete open subset of $\omega_1^{\mathcal{S}} = (\omega_1, \tau_{\mathcal{S}})$ and $L(\omega_1)$ is a discrete closed subset of $\omega_1^{\mathcal{S}}$. The topology $\tau_{\mathcal{S}}$ is finer than the order topology. The relatively compact subsets of $\omega_1^{\mathcal{S}}$ are those which are almost contained in a finite union of ladders s_α (with “almost” meaning “except for a finite set”). The sequence $(s_\alpha^n)_{n \in \omega}$ converges to α in $\omega_1^{\mathcal{S}}$ and a map f defined in $\omega_1^{\mathcal{S}}$ (taking values in an arbitrary space) is continuous if and only if $(f(s_\alpha^n))_{n \in \omega}$ converges to $f(\alpha)$, for all $\alpha \in L(\omega_1)$. The space $\omega_1^{\mathcal{S}}$ is locally compact Hausdorff of height 2 and we denote by $K_{\mathcal{S}} = \omega_1 \cup \{\infty\}$ its one-point compactification (which is compact Hausdorff of height 3). Since every point of $\omega_1^{\mathcal{S}}$ has a countable neighborhood and compact subsets of $\omega_1^{\mathcal{S}}$ are countable, a continuous map $\phi : K_{\mathcal{S}} \rightarrow K_{\mathcal{S}}$ with $\phi(\infty) \neq \infty$ has countable range. We are thus interested in maps ϕ fixing the point ∞ .

Proposition 1. *Let $\phi : K_{\mathcal{S}} \rightarrow K_{\mathcal{S}}$ be a continuous map with $\phi(\infty) = \infty$. Then there exists a club $F \subset \omega_1$ such that $\phi(\alpha) = \alpha$ or $\phi(\alpha) = \infty$, for all $\alpha \in F$.*

Proof. For $\alpha \in \omega_1$, the level set $\phi^{-1}(\alpha)$ is a compact subset of $\omega_1^{\mathcal{S}}$ and therefore countable. Thus the Pressing Down Lemma implies that the set $\{\alpha \in \omega_1 : \phi(\alpha) < \alpha\}$ is nonstationary. Let F_1 be a club disjoint from the latter. Consider the map $\psi : \omega_1 \rightarrow \omega_1$ given by $\psi(\alpha) = \phi(\alpha)$ if $\phi(\alpha) \neq \infty$ and $\psi(\alpha) = 0$ otherwise. By standard arguments, the set F_2 of ordinals α invariant by ψ (i.e., $\beta < \alpha$ implies $\psi(\beta) < \alpha$) is club. For $\alpha \in F_2 \cap L(\omega_1)$, $\phi[s_\alpha]$ is contained in the closed set $[0, \alpha] \cup \{\infty\}$ and therefore also $\phi(\alpha)$ is in $[0, \alpha] \cup \{\infty\}$. The conclusion is obtained by taking $F = F_1 \cap F_2 \cap L(\omega_1)$. \square

One might ask whether Proposition 1 can be strengthened, providing the result that if ϕ has uncountable range then its set of fixed points contains a club. The following example shows that this is not the case.

2.1. Example. Let $\phi : K_{\mathcal{S}} \rightarrow K_{\mathcal{S}}$ be a map whose restriction to $S(\omega_1)$ is an injection into $L(\omega_1)$ and that maps every element of $L(\omega_1) \cup \{\infty\}$ to ∞ . For limit α , the sequence $(\phi(s_\alpha^n))_{n \in \omega}$ is injective and contained in $L(\omega_1)$; therefore it converges to $\infty = \phi(\alpha)$. This proves the continuity of ϕ at the points of ω_1 . It is easy to see that a map $\psi : K_{\mathcal{S}} \rightarrow K_{\mathcal{S}}$ with $\psi(\infty) = \infty$ is continuous at ∞ if and only if $\psi^{-1}(\alpha)$ is relatively compact for all $\alpha \in \omega_1$ and also $\psi^{-1}[s_\alpha]$ is relatively compact for all $\alpha \in L(\omega_1)$. It follows from this criterion that ϕ is continuous at ∞ .

Proposition 1 says that continuous self-maps of K_S (fixing ∞) are highly constrained in a club subset F : namely every point of F is either fixed or mapped to ∞ . We now show that on the nonstationary set $L(\omega_1) \setminus F$ a continuous self-map of K_S is more or less arbitrary.

Theorem 2. *Let $\xi : A \rightarrow L(\omega_1)$ be an injective map defined in a nonstationary subset A of $L(\omega_1)$. Then ξ extends to a continuous self-map ϕ of K_S that maps ∞ and every element of $L(\omega_1) \setminus A$ to ∞ .*

Proof. We claim that the ladders s_α , $\alpha \in A$, can be made disjoint by removing a finite subset from each of them. Obviously, this will not change τ_S . Note that the ladders are already almost disjoint. Let $F \subset \omega_1$ be a club disjoint from A and, for $\alpha \in A$, let $\lambda(\alpha)$ be the largest element of $F \cup \{0\}$ below α . Then $\lambda : A \rightarrow \omega_1$ is regressive (i.e., $\lambda(\alpha) < \alpha$, for all $\alpha \in A$) and cofinal (i.e., for all $\alpha \in \omega_1$, there exists $\beta_0 \in \omega_1$, such that, for all $\beta \in A$, $\beta \geq \beta_0$ implies $\lambda(\beta) \geq \alpha$). For $\alpha \in A$, remove the finite number of elements of s_α below $\lambda(\alpha)$. Consider the binary relation R on A defined by $(\alpha, \beta) \in R \Leftrightarrow s_\alpha \cap s_\beta \neq \emptyset$. The fact that λ is cofinal implies that, for each $\alpha \in A$, only a countable number of elements of A are R -related to α . Hence the equivalence classes defined by the equivalence relation spanned by R are also countable. The proof of the claim is now obtained by recalling that a countable number of almost disjoint sets can be made disjoint by removing a finite subset from each of them.

The ladders s_α , $\alpha \in A$, are now assumed to be disjoint. Define ϕ by mapping α and each element of s_α to $\xi(\alpha)$, for $\alpha \in A$; map everything else to ∞ . The continuity of ϕ at the points of A is clear. For $\alpha \in L(\omega_1) \setminus A$, the sequence $(\phi(s_\alpha^n))_{n \in \omega}$ is contained in $L(\omega_1) \cup \{\infty\}$ and has no constant subsequence contained in $L(\omega_1)$. Therefore, it converges to $\infty = \phi(\alpha)$. Finally, the continuity of ϕ at ∞ follows from the criterion explained in Example 2.1. \square

In the statement of Theorem 2 the assumptions on ξ can be weakened: namely, it suffices to assume that $\xi : A \rightarrow L(\omega_1) \cup \{\infty\}$ be a map with $\xi^{-1}(\alpha)$ finite, for all $\alpha \in L(\omega_1)$. Note that this weaker assumption on ξ is necessary: if $\alpha \in L(\omega_1)$ then $\phi^{-1}(\alpha)$ is a compact subset of ω_1^S and hence $\phi^{-1}(\alpha) \cap L(\omega_1)$ must be finite.

2.1. Operators on $C(K_S)$. We now discuss the bounded operators on the Banach space $C(K_S)$ of continuous real-valued maps on K_S . Given a compact Hausdorff space K and $g \in C(K)$, we denote by M_g the multiplication operator $f \mapsto fg$ on $C(K)$. If g belongs to the space $\mathfrak{B}(K)$ of real-valued bounded Borel functions on K , we denote by M_g^* the operator on $C(K)^*$ defined by $\mu \mapsto \int g d\mu$. (When g is continuous, M_g^* is indeed the adjoint of M_g .) In [2], an operator T on $C(K)$ is called a *weak multiplication* if $T + M_g$ is weakly compact for some $g \in C(K)$ and it is called a *weak multiplier* if $T^* + M_g^*$ is weakly compact, for some $g \in \mathfrak{B}(K)$. Obviously every weak multiplication is a weak multiplier and, in [2], the space $C(K)$ is said to

have *few operators* if every operator is a weak multiplier. If K has a non-trivial convergent sequence (for instance, if K is infinite and scattered) then $C(K)$ cannot have few operators in this sense: namely, in this case $C(K)$ has a complemented copy of c_0 and thus is isomorphic to its closed hyperplanes. But this is impossible if $C(K)$ has few operators ([2, Theorem 3.2]). However, in [3], it is presented (under \clubsuit) an example of a nonmetrizable scattered compact space K (of infinite height) for which every operator on $C(K)$ is a constant multiple of the identity plus an operator of separable range. Having this in mind, for a given compact Hausdorff space K , one can consider the question of whether every operator T on $C(K)$ satisfies one of the following conditions:

- (a) $T + M_g$ has separable range, for some $g \in C(K)$;
- (b) $T^* + M_g^*$ has separable range, for some $g \in \mathfrak{B}(K)$;
- (c) the restriction of $(T^* + M_g^*)^*$ to $C(K)$ has separable range, for some $g \in \mathfrak{B}(K)$.

Condition (a) is a natural modification of the notion of weak multiplication and (b), (c) are natural candidates for modifications of the notion of weak multiplier. Note that (a) implies (c), but not (b). We now prove that, if $K = K_S$ and ϕ is defined as in Example 2.1, then the composition operator \mathcal{C}_ϕ does not satisfy either (b) or (c) (and hence does not satisfy (a)). The fact that $T = \mathcal{C}_\phi$ does not satisfy (b) follows from the observation that $T^* + M_g^*$ maps $\{\delta_\alpha : \alpha \in S(\omega_1)\}$ to an uncountable discrete set, where δ_α denotes the delta-measure supported at α .

To prove that $T = \mathcal{C}_\phi$ does not satisfy (c), note first that the restriction of $(T^* + M_g^*)^*$ to $C(K_S)$ is identified with $T + M_g : C(K_S) \rightarrow \mathfrak{B}(K_S)$, where $\mathfrak{B}(K_S)$ is identified with a subspace of $C(K_S)^{**}$ in the natural way. Assuming that $\mathcal{C}_\phi + M_g$ has separable range, we prove first that g must vanish outside a countable set. If there were uncountably many $\alpha \in S(\omega_1)$ with $g(\alpha) \neq 0$, there would exist $\varepsilon > 0$ and an uncountable subset A of $S(\omega_1)$ with $|g(\alpha)| \geq \varepsilon$, for all $\alpha \in A$. Then $\mathcal{C}_\phi + M_g$ would map $\{\chi_{\{\alpha\}} : \alpha \in A\}$ to an uncountable discrete set (where χ denotes characteristic function). Similarly, if there were uncountably many $\alpha \in L(\omega_1)$ with $g(\alpha) \neq 0$, there would exist $\varepsilon > 0$ and an uncountable subset A of $L(\omega_1)$ with $|g(\alpha)| \geq \varepsilon$, for all $\alpha \in A$. Then $\mathcal{C}_\phi + M_g$ would map $\{\chi_{s_\alpha \cup \{\alpha\}} : \alpha \in A\}$ to an uncountable discrete set. We have thus proven that the support of g , $\text{supp } g$, must be countable. This implies that M_g has separable range, since it factorizes through the restriction map $C(K_S) \rightarrow C(\text{supp } g)$ and $C(\text{supp } g)$ is separable. Finally, the fact that M_g has separable range implies that \mathcal{C}_ϕ does as well. This is a contradiction, because $\mathcal{C}_\phi[C(K_S)] \equiv C(\phi[K_S])$ and $\phi[K_S]$ contains an uncountable discrete set.

REFERENCES

- [1] A. V. Arkhangel'skii, Topological Function Spaces, Kluwer, 1992.

- [2] P. Koszmider, A survey on Banach spaces $C(K)$ with few operators, *RACSAM* 104 (2), 2010, pgs. 309—326.
- [3] P. Koszmider & P. Zieliński, Complementation and Decompositions in some weakly Lindelöf Banach spaces, *J. Math. Anal. Appl.* 376, 2011, pgs. 329–341.
- [4] R. Pol, A function space $C(X)$ which is weakly Lindelöf but not weakly compactly generated, *Studia Math.* T. LXIV, 1979, pgs. 279—285.

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